Дз 3

№ 2

$$\begin{cases} u_t = u_{xx} + \sin(3\pi x)\cdot e^{-9\pi^2 t} & 0<x<1, \quad t>0 \\ u(0,t)=0 & \\ u(1,t)=0 & t>0 \\ u(x,0)=0 & 0\le x\le 1 \end{cases}$$

I $X_n = \sin\sqrt{\lambda_n}\,x \qquad \lambda_n = \left(\frac{\pi n}{l}\right)^2 = (\pi n)^2 \quad n = 1, 2, \dots$

II $f(x,t) = \sum_{n=1}^{\infty} f_n\cdot\sin(\pi n x) = \sin(3\pi x)\cdot e^{-9\pi^2 t}$

$f_3 = e^{-9\pi^2 t}, \quad f_i = 0$ при $i\neq 3 \quad i\in\mathbb{N}$

III $\begin{cases} T_n' + \lambda_n T_n = f_n \\ T_n(0) = 0 \end{cases}$

$T_n = e^{-(\pi n)^2 t}\cdot c_n(t)$

$e^{-(\pi n)^2 t}\cdot c_n'(t) = f_n$

1) $e^{-9\pi^2 t}\, c_3'(t) = e^{-9\pi^2 t}$

$\Rightarrow c_3'(t) = 1 \Rightarrow c_3(t) = t + \tilde{c}_3$

2) $e^{-(n\pi)^2 t}\, c_n'(t) = 0$

$\Rightarrow c_n'(t) = 0$ при $n\neq 3 \; n\in\mathbb{N}$

$\Rightarrow c_n(t) = \tilde{c}_n$

$T_n(0) = c_n(0) = \tilde{c}_n = 0$

$\Rightarrow T_3 = t e^{-9\pi^2 t}$

$T_n = 0, \quad n\neq 3 \quad n\in\mathbb{N}$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} X_n T_n = \sin(3\pi x)\cdot e^{-9\pi^2 t}\cdot t}$$

ДЗ 3

№ 3

$$\begin{cases} u_t = a^2 u_{xx} + f_0 & 0<x<l \quad t>0 \quad f_0 = const \\ u(0,t)=0 & \\ u(l,t)=0 & t>0 \\ u(x,0)=0 & 0 \le x \le l \end{cases}$$

I $\quad X_n = \sin(\sqrt{\lambda_n}\, x) \qquad \lambda_n = \left(\frac{\pi n}{l}\right)^2 \qquad n = 1,2,\ldots$

II $\quad f(x,t) = \sum_{n=1}^{\infty} f_n X_n = f_0$

$$f_n = \frac{2}{l}\int_0^l f_0 \sin\left(\frac{\pi n}{l}x\right)dx = \frac{2}{l} f_0\left(-\frac{\cos\left(\frac{\pi n}{l}x\right)}{\frac{\pi n}{l}}\Big|_0^l\right) = \frac{2f_0}{l\,\frac{\pi n}{l}}\left(1-(-1)^n\right) =$$

$$= \begin{cases} \frac{4f_0}{\pi n}, & n = 2k+1 \\ 0, & n = 2k \end{cases}$$

III $\quad \begin{cases} T_n' = -\lambda_n a^2 T_n + f_n \\ T_n(0) = 0 \end{cases}$

1) $\overset{n=2k}{f_n = 0} \Rightarrow$

$T_n = e^{-\lambda_n a^2 t} C_n$

$T_n(0) = C_n = 0$

$\Rightarrow T_n = 0$

2) $\overset{n=2k+1}{f_n = \frac{4f_0}{\pi n}}$

$T_n = e^{-\lambda_n a^2 t} \cdot C_n(t)$

$e^{-\lambda_n a^2 t} C_n'(t) = f_n$

$C_n(t) = \int f_n \cdot e^{+\lambda_n a^2 t}\,dt$

$C_n(t) = \frac{f_n e^{\lambda_n a^2 t}}{\lambda_n a^2} + \tilde{C}_n$

$T_n(0) = C_n(0) = \frac{f_n}{\lambda_n a^2} + \tilde{C}_n = 0$

$\Rightarrow \tilde{C}_n = -\frac{f_n}{\lambda_n a^2}$

$\Rightarrow T_n(t) = e^{-\lambda_n a^2 t}\frac{f_n}{\lambda_n a^2}\left(e^{\lambda_n a^2 t} - 1\right) =$

$= \frac{f_n}{\lambda_n a^2}\left(1 - e^{-\lambda_n a^2 t}\right) = \frac{4f_0}{\pi n \left(\frac{\pi n}{l}\right)^2 a^2}\left(1 - e^{-\left(\frac{\pi n}{l}\right)^2 a^2 t}\right) =$

$= \frac{4f_0 \cdot l^2}{(\pi n)^3 a^2}\left(1 - e^{-\left(\frac{\pi n a}{l}\right)^2 t}\right)$

$$\boxed{\begin{aligned} u(x,t) &= \left[\sum_{k=0}^{\infty} \sin\left(\frac{\pi(2k+1)}{l}x\right)\cdot\frac{1}{(2k+1)^3}\left(1 - e^{-\left(\frac{\pi(2k+1)a}{l}\right)^2 t}\right)\right]\cdot\frac{4f_0 l^2}{\pi^3 a^2} \\ \lim_{t\to\infty} u(x,t) &= \sum_{k=0}^{\infty}\frac{\sin\left(\frac{\pi(2k+1)}{l}x\right)}{(2k+1)^3}\cdot\frac{4f_0 l^2}{a^2\pi^3} \end{aligned}}$$

Дз 3

№ 4

$$\begin{cases} u_t = a^2 u_{xx} + f_0 & 0<x<l \quad t>0 \quad f_0 = const \\ u_x(0,t) = 0 & \\ u_x(l,t) = 0 & t>0 \\ u(x,0) = 0 & 0 \le x \le l \end{cases}$$

$X_n = \cos(\sqrt{\lambda_n}\, x) \qquad \lambda_n = \left(\frac{\pi n}{l}\right)^2$

$X_0 = 1 \qquad \lambda_0 = 0$

$f(x,t) = \sum_{n=0}^{\infty} \varphi_n X_n = f_0 = \varphi_0 + \sum_{n=1}^{\infty} \varphi_n \cos(\sqrt{\lambda_n}\, x)$ обозначила через φ_k, потому что f_0 уже занято :)

$\varphi_0 = f_0 \qquad \varphi_n = 0 \qquad n \in \mathbb{N}$

1) $T_0' = -0 \cdot a^2 T_0 + f_0$

$$\begin{cases} T_0' = f_0 \\ T(0) = 0 \end{cases}$$

$T_0 = f_0 t + C$

$T_0(0) = C = 0$

$\Rightarrow T_0(t) = f_0 t$

2) $T_n' = -\lambda_n a^2 T_n + 0$

$T_n = C_n e^{-\lambda_n a^2 t}$

$T_n(0) = 0 = C_n$

$n \in \mathbb{N}$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} T_n X_n = f_0 t}$$

$$\boxed{\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} f_0 t = \infty,} \begin{cases} +\infty, \text{ если } f_0 > 0 \\ -\infty, \text{ если } f_0 < 0 \end{cases}$$

№ 6

$$\begin{cases} u_t = a^2 u_{xx} + f_0 & 0<x<l, \quad t>0 \quad f_0 = const \\ u_x(0,t) = 0 & \\ u(l,t) = 0 & t>0 \\ u(x,0) = 0 & 0 \le x \le l \end{cases}$$

I $X_n = \cos(\sqrt{\lambda_n}\, x) \qquad \lambda_n = \left(\frac{\pi + 2\pi n}{2l}\right)^2$

II $\varphi_n = \frac{2}{l} \cdot \int_0^l f_0 \cos\left(\frac{\pi + 2\pi n}{2l} x\right) dx = \frac{2 f_0}{l} \left.\frac{\sin\left(\frac{\pi+2\pi n}{2l} x\right)}{\frac{\pi + 2\pi n}{2l}}\right|_0^l = \frac{4 f_0}{\pi + 2\pi n} (-1)^n$

III $\begin{cases} T_n' = -a^2 T_n \lambda_n + \varphi_n \\ T(0) = 0 \end{cases}$

$T_n = e^{-a^2 t \lambda_n} \cdot \int_0^t e^{a^2 \tau \lambda_n} \varphi_n(\tau)\, d\tau = e^{-a^2 t \left(\frac{\pi + 2\pi n}{2l}\right)^2} \cdot \left(\frac{e^{a^2 t \left(\frac{\pi+2\pi n}{2l}\right)^2} - 1}{a^2 \left(\frac{\pi + 2\pi n}{2l}\right)^2}\right) \cdot \varphi_n$

$$u(x,t) = \frac{16 f_0 l^2}{a^2 \pi^3} \sum_{n=0}^{\infty} \frac{\cos\left(\frac{\pi + 2\pi n}{2l} x\right)\left(1 - e^{-a^2 t \left(\frac{\pi + 2\pi n}{2l}\right)^2}\right)}{(1+2n)^3}$$

Дз 3

№ 10

$$\begin{cases} u_t = u_{xx} + 2 + 2\sin 5x & 0 < x < \frac{\pi}{2} \quad t > 0 \\ u(0,t) = 2t & \\ u_x(\frac{\pi}{2}, t) = 0 & t > 0 \\ u(x,0) = 0 & 0 \le x \le \frac{\pi}{2} \end{cases}$$

Ненулевые граничные условия!

Делаем замену:

$u = v + 0 \cdot x + 2t$

$$\begin{cases} v_t = v_{xx} + 2\sin 5x \\ v(0,t) = 0 \\ v_x(\frac{\pi}{2}; t) = 0 \\ v(x,0) = 0 \end{cases}$$

I $X_n = \sin(\sqrt{\lambda_n}\, x)$ $\qquad \lambda_n = \left(\frac{\pi + 2\pi n}{2l}\right)^2 = (1+2n)^2$

II $f(x,t) = \sin 5x = \sum_{n=0}^{\infty} f_n \cdot \sin(1+2n)x$

$\Rightarrow \quad f_2 = 2 \quad f_i = 0 \quad i \neq 2 \;\; i \in \mathbb{N} \cup \{0\}$

III $T_i = 0 \quad i \neq 2, \; i \in \mathbb{N} \cup \{0\}$

$$T_2 = \int_0^t e^{-a^2(t-\tau)\lambda_2}\, 2\, d\tau = +\frac{2}{a^2\lambda_2}\left(1 - e^{-a^2 t \lambda_2}\right) = \frac{2}{25}\left(1 - e^{-25t}\right)$$

$$u(x,t) = \frac{2\sin 5x}{25} \cdot \left(1 - e^{-25t}\right) + 2t$$